

# Floating-point numbers

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**Laurent Bartholdi**

MPFR- and CXSC-based library for GAP

**Laurent Bartholdi** Email: [laurent.bartholdi@gmail.com](mailto:laurent.bartholdi@gmail.com)  
Homepage: <http://www.uni-math.gwdg.de/laurent/>

**Address:** Mathematisches Institut  
Bunsenstraße 3-5  
D-37073 Göttingen  
Germany

## **Abstract**

This document describes the package `Float`, which implements in `GAP` arbitrary-precision floating-point numbers.

For comments or questions on `Float` please contact the author.

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# Chapter 1

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## Chapter 2

# Float package

### 2.1 A sample run

The extended floating-point capabilities of GAP are installed by loading the package via `LoadPackage("float");` and selecting new floating-point handlers via `SetFloats(MPFR)`, `SetFloats(MPFI)`, `SetFloats(MPC)` or `SetFloats(CXSC)`, depending on whether high-precision real, interval or complex arithmetic are desired, or whether a fast package containing all four real/complex element/interval arithmetic is desired:

Example

```
gap> LoadPackage("float");
Loading FLOAT 0.7.0 ...
true
gap> SetFloats(MPFR); # floating-point
gap> x := 4*Atan(1.0);
.314159e1
gap> Sin(x);
.169569e-30
gap> SetFloats(MPFR,1000); # 1000 bits
gap> x := 4*Atan(1.0);
.314159e1
gap> Sin(x);
.125154e-300
gap> String(x,300);
".3141592653589793238462643383279502884197169399375105820974944592307816406286\
208998628034825342117067982148086513282306647093844609550582231725359408128481\
117450284102701938521105559644622948954930381964428810975665933446128475648233\
78678316527120190914564856692346034861045432664821339360726024914127e1"
gap>
gap> SetFloats(MPFI); # intervals
gap> x := 4*Atan(1.0);
.314159e1(99)
gap> AbsoluteDiameter(x); Sup(x); Inf(x);
.100441e-29
.314159e1
.314159e1
gap> Sin(x);
-.140815e-29(97)
gap> 0.0 in last;
```

```
true
gap> 1.0; # exact representation
.1e1(inf)
gap> IncreaseInterval(last,0.001); # now only 8 significant bits
.1e1(8)
gap> IncreaseInterval(last,-0.002); # now becomes empty
\emptyset
gap> MinimalPolynomial(Rationals,Sqrt(2.0));
-2*x_1^2+1
gap> Cyc(last);
E(8)-E(8)^3
gap>
gap> SetFloats(MPC); # complex numbers
gap> z := 5.0-1.0i;
.5e1-.1e1i
gap> (1+1.0i)*last^4*(239+1.0i);
.228488e6
gap> Exp(6.2835i);
.1e1+.314693e-3i
```

## Chapter 3

# Polynomials

### 3.1 The Floats pseudo-field

Polynomials with floating-point coefficients may be manipulated in GAP; though they behave, in subtle ways, quite differently than polynomials over rings.

The "pseudo-field" of floating-point numbers is an object in GAP, called `FLOAT_PSEUDOFIELD`. (It is not really a field, e.g. because addition of floating-point numbers is not associative). It may be used to create indeterminates, for example as

Example

```
gap> x := Indeterminate(FLOAT_PSEUDOFIELD, "x");
x
gap> 2*x^2+3;
2.0*x^2+3.0
gap> Value(last, 10);
203.0
```

### 3.2 Roots of polynomials

The Jenkins-Traub algorithm has been implemented, in arbitrary precision for MPFR and MPC.

Furthermore, CXSC can provide complex enclosures for the roots of a complex polynomial.

### 3.3 Finding integer relations

The PSLQ algorithm has been implemented by Steve A. Linton, as an external contribution to Float. This algorithm receives as input a vector of floats  $x$  and a required precision  $\epsilon$ , and seeks an integer vector  $v$  such that  $|x \cdot v| < \epsilon$ . The implementation follows quite closely the original article [BB01].

#### 3.3.1 PSLQ

▷ `PSLQ(x, epsilon[, gamma])` (function)

▷ `PSLQ_MP(x, epsilon[, gamma[, beta]])` (function)

**Returns:** An integer vector  $v$  with  $|x \cdot v| < \epsilon$ .

The PSLQ algorithm by Bailey and Broadhurst (see [BB01]) searches for an integer relation between the entries in  $x$ .

$\beta$  and  $\gamma$  are algorithm tuning parameters, and default to  $4/10$  and  $2/\sqrt{3}$  respectively.

The second form implements the "Multi-pair" variant of the algorithm, which is better suited to parallelization.

Example

```
gap> PSLQ([1.0, (1+Sqrt(5.0))/2], 1.e-2);  
[ 55, -34 ] # Fibonacci numbers  
gap> RootsFloat([1, -4, 2]*1.0);  
[ 0.292893, 1.70711 ] # roots of 2x^2-4x+1  
gap> PSLQ(List([0..2], i->last[1]^i), 1.e-7);  
[ 1, -4, 2 ] # a degree-2 polynomial fitting well
```

### 3.4 LLL lattice reduction

A faster implementation of the LLL lattice reduction algorithm has also been implemented. It is accessible via the commands `FPLLLReducedBasis(m)` and `FPLLLShortestVector(m)`.



## Chapter 4

# Implemented packages

### 4.1 MPFR

#### 4.1.1 IsMPFRFloat

- ▷ IsMPFRFloat (filter)
- ▷ TYPE\_MPFR (global variable)

The category of floating-point numbers.

Note that they are treated as commutative and scalar, but are not necessarily associative.

### 4.2 MPFI

#### 4.2.1 IsMPFIFloat

- ▷ IsMPFIFloat (filter)
- ▷ TYPE\_MPFI (global variable)

The category of intervals of floating-point numbers.

Note that they are treated as commutative and scalar, but are not necessarily associative.

### 4.3 MPC

#### 4.3.1 IsMPCFloat

- ▷ IsMPCFloat (filter)
- ▷ TYPE\_MPC (global variable)

The category of intervals of floating-point numbers.

Note that they are treated as commutative and scalar, but are not necessarily associative.

## 4.4 CXSC

### 4.4.1 IsCXSCReal

▷ IsCXSCReal	(filter)
▷ IsCXSCComplex	(filter)
▷ IsCXSCInterval	(filter)
▷ IsCXSCBox	(filter)
▷ TYPE_CXSC_RP	(global variable)
▷ TYPE_CXSC_CP	(global variable)
▷ TYPE_CXSC_RI	(global variable)
▷ TYPE_CXSC_CI	(global variable)

The category of floating-point numbers.

Note that they are treated as commutative and scalar, but are not necessarily associative.

## 4.5 FPLLL

### 4.5.1 FPLLLReducedBasis

▷ FPLLLReducedBasis( $m$ )	(operation)
----------------------------	-------------

**Returns:** A matrix spanning the same lattice as  $m$ .

This function implements the LLL (Lenstra-Lenstra-Lovász) lattice reduction algorithm via the external library `fpLLL`.

The result is guaranteed to be optimal up to 1%.

### 4.5.2 FPLLLShortestVector

▷ FPLLLShortestVector( $m$ )	(operation)
------------------------------	-------------

**Returns:** A short vector in the lattice spanned by  $m$ .

This function implements the LLL (Lenstra-Lenstra-Lovász) lattice reduction algorithm via the external library `fpLLL`, and then computes a short vector in this lattice.

The result is guaranteed to be optimal up to 1%.

# References

- [BB01] D. H. Bailey and D. J. Broadhurst. Parallel integer relation detection: techniques and applications. *Math. Comp.*, 70(236):1719–1736 (electronic), 2001. [7](#)

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